# 12.3 DOT PRODUCT

Ask for definition in Linear Algebra

**Definition:**

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \langle b_1, b_2, b_3 \rangle$$

Then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

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**Properties** (Show slide)

1) $\vec{a} \cdot \vec{a} = |a|^2$, since (ask Students)

$$\vec{a} \cdot \vec{a} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle =$$

$$a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$$

An alternative definition of the dot product is given by the following theorem:

<u>Theorem.</u>

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta,$$

where $\theta$ is the angle in between $\vec{a}$ and $\vec{b}$

$0 \le \theta \le \pi$.

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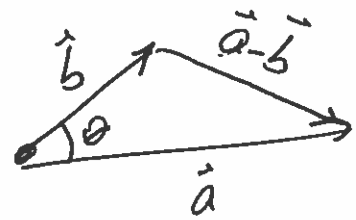
<u>Proof.</u>— The key idea of this proof is the application of the Law of cosines to the triangle




$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta \qquad (2.1)$$

Now, $$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 - 2\,\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

Substitution into (2.1)

$$\cancel{|\vec{a}|^2} - 2\vec{a}.\vec{b} + \cancel{|\vec{b}|^2} = \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \quad \cancel{2}\vec{a}.\vec{b} = \cancel{2}|\vec{a}||\vec{b}|\cos\theta$$

or

$$\boxed{\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta} \qquad (3.1)$$

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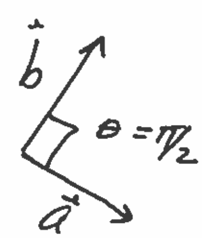
An inmediate consequence of formula (3.1) is

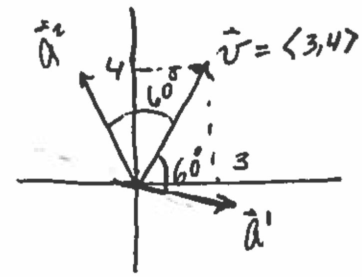
$$\boxed{\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}} \qquad (3.2)$$

Formula to find angle $\theta$ between vectors $\vec{a}$ and $\vec{b}$.

Obviously if the angle $\theta$ between two vectors is $\theta = \pi/2$, then

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\pi/2 = |\vec{a}||\vec{b}|\,0 = 0$$






# Section 12.3

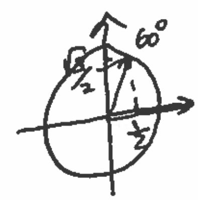
## Exercise # 28

Find two unit vectors that make an angle of $60°$ with $\vec{v} = \langle 3,4 \rangle$.

Two unknowns

<u>Answ</u>: $\vec{a} = \langle \overset{?}{a_1}, \overset{?}{a_2} \rangle$ Such that $\boxed{|\vec{a}|^2 = a_1^2 + a_2^2 = 1}$. (1)

Also,




$$\frac{1}{2} = \cos \pi/3 = \frac{\vec{a} \cdot \vec{v}}{|\vec{a}||\vec{v}|}$$

or $$\frac{1}{2} = \frac{3a_1 + 4a_2}{(1)(5)} \Leftrightarrow \boxed{3a_1 + 4a_2 = \frac{5}{2}} \qquad (2)$$

From (2) $$a_2 = \frac{5}{8} - \frac{3}{4}a_1$$

Subst. into (1) $$a_1^2 = 1 - \left(\frac{5}{8} - \frac{3}{4}a_1\right)^2$$

$$\frac{25}{64} - \frac{30}{32}a_1 + \frac{9}{16}a_1^2 + a_1^2 - 1 = 0$$

$$\begin{array}{r} 64 \\ 25 \\ \hline 39 \end{array}$$

$$\Leftrightarrow \frac{25}{16}a_1^2 - \frac{30}{32}a_1 - \frac{39}{64} = 0$$

$$\Leftrightarrow \boxed{100a_1^2 - 60a_1 - 39 = 0}$$

$$\boxed{100\,a_1^2 - 60a_1 - 39 = 0}$$

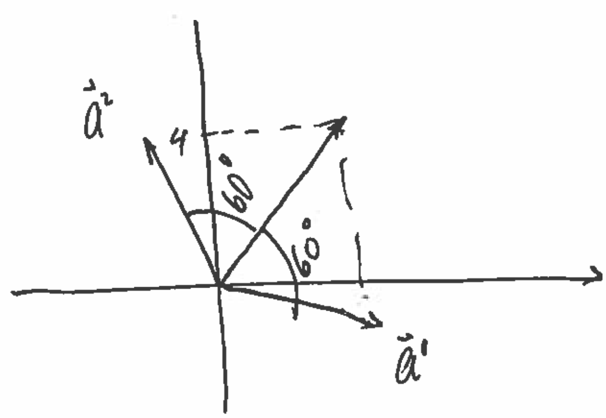
$$a_1 = \frac{3 \pm 4\sqrt{3}}{10}$$

If $a_1 = \dfrac{3+4\sqrt{3}}{10} \Rightarrow a_2 = \dfrac{4-3\sqrt{3}}{10}$

or $\boxed{\vec{a}^1 = \langle 0.9928, -0.1196 \rangle}$

If $a_1 = \dfrac{3-4\sqrt{3}}{10} \Rightarrow a_2 = \dfrac{4+3\sqrt{3}}{10}$

or $\boxed{\vec{a}^2 = \langle -0.3928, 0.9196 \rangle}$.



**Definition**

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are orthogonal if the angle between them is $\theta = \pi/2$.

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Since $\vec{0}$ (vector) is considered to be orthogonal to all vectors, we have the alternative definition

**Definition**

Two vectors $\vec{a}$ and $\vec{b}$ ($\mathbb{R}^2$ or $\mathbb{R}^3$) are orthogonal if $\vec{a}\cdot\vec{b} = 0$.

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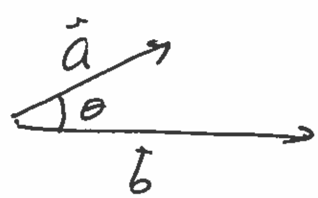
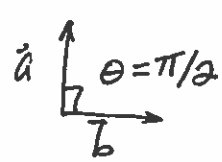
**Possibilities:** $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$

a)



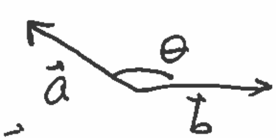

$\vec{a}\cdot\vec{b} > 0$, $0 \leq \theta < \pi/2$

b)

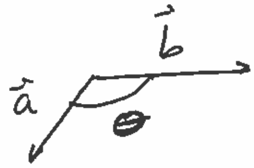


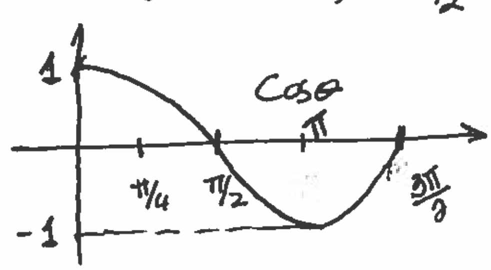

$\vec{a}\cdot\vec{b} = 0$

c)

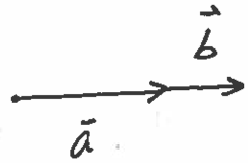



$\vec{a}\cdot\vec{b} < 0$, $\pi/2 < \theta \leq \pi$



d) If $\theta = 0$




$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|.$$

e) If $\theta = \pi$


$$\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|.$$

## Direction Angles and Direction Cosines.

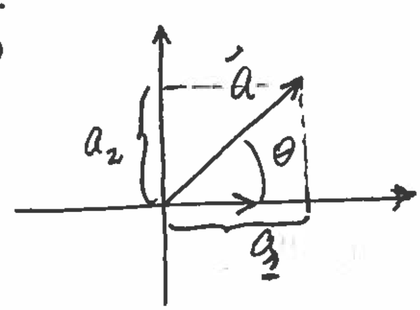
**Definition** Given a nonzero vector $\vec{a}$

a) The angles $\alpha, \beta, \gamma$ that $\vec{a}$ makes with the positive axes $x, y,$ and $z$, respectively are called the direction angles of $\vec{a}$.

b) The cosines of the direction angles of $\vec{a}$ are called the direction cosines of $\vec{a}$.

Show slides.

In $\mathbb{R}^2$,

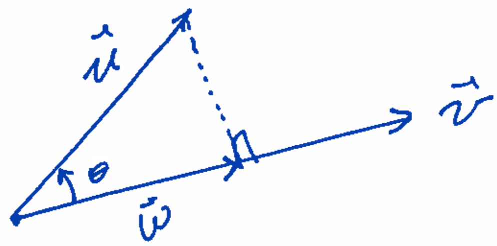



$$\vec{a}\cdot\hat{\imath} = |\vec{a}||\hat{\imath}|\cos\theta$$
$$= |\vec{a}|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\hat{\imath}}{|\vec{a}|} = \frac{a_1}{|\vec{a}|}$$

$$\vec{a}\cdot\hat{\imath} = \langle a_1, a_2\rangle \cdot \langle 1, 0\rangle = a_1$$

# Vector Projection and Scalar Projection



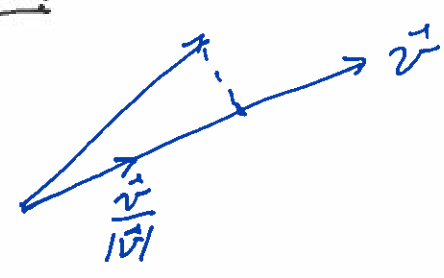

**Def.**–
The vector $\vec{\omega}$ obtained by intersecting a perpendicular line from terminal point of $\vec{u}$ to the line passing along $\vec{v}$ is called

$$\boxed{\vec{\omega} = \text{Vector projection of } \vec{u} = \text{proj}_{\vec{v}} \vec{u}}$$

Also, the component of $\vec{u}$ along $\vec{v}$ is called the "Scalar projection of $\vec{u}$ onto $\vec{v}$"

$$\boxed{|\vec{\omega}| = |\vec{u}| \cos\theta = \text{Comp}_{\vec{v}} \vec{u}}$$

In terms of dot product

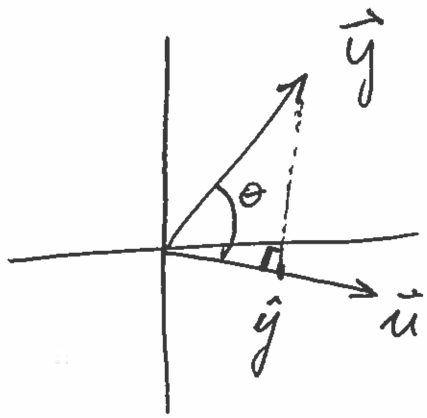



$$\boxed{\text{proj}_{\vec{v}} \vec{u} = \left(\vec{u} \cdot \frac{\vec{v}}{|\vec{v}|}\right) \frac{\vec{v}}{|\vec{v}|}}$$

$$\boxed{\text{Comp}_{\vec{v}} \vec{u} = u \cdot \frac{\vec{v}}{|\vec{v}|} = \frac{|\vec{u}|\,|\vec{v}|}{|\vec{v}|} \cos\theta = |\vec{u}| \cos\theta}$$

Find the orthogonal projection of $\vec{y}$ onto $\vec{u}$ if the angle between these two vectors is $\theta$.

In $\mathbb{R}^2$ (Completely Similar in $\mathbb{R}^3$)




From elementary trigonometry, the of $\hat{y}$ is

$$|\hat{y}| = |\vec{y}| \cos\theta$$

Also, since $\hat{y}$ is in the direction of $\vec{u}$

$$\hat{y} = |\hat{y}| \frac{\vec{u}}{|\vec{u}|} = \left(|\vec{y}| \cos\theta\right) \frac{\vec{u}}{|\vec{u}|}$$

↳ unit vector in the direction of $\vec{u}$

Noticing that

$$\vec{y} \cdot \frac{\vec{u}}{|\vec{u}|} = |\vec{y}| \, \frac{|\vec{u}|}{|\vec{u}|} \cos\theta = |\vec{y}| \cos\theta$$

we obtain

$$\text{proj}_{\vec{u}} \vec{y} = \hat{y} = \left(\vec{y} \cdot \frac{\vec{u}}{|\vec{u}|}\right) \frac{\vec{u}}{|\vec{u}|}$$

Again,

$$\text{proj}_{\vec u}\, \hat y = \left(\vec y \cdot \frac{\vec u}{|\vec u|}\right) \frac{\vec u}{|\vec u|}$$

Thus, to obtain the projection of $\vec y$ onto $\vec u$

1) Dot product $\vec y$ and the unit vector $\frac{\vec u}{|\vec u|}$ in the direction of $\vec u$.
   This is the component of $\vec y$ in the direction of $\vec u$.

2) Construct the projection $\hat y$ of $\vec y$ onto $\vec u$ by multiplying the scalar $\left(\vec y \cdot \frac{\vec u}{|\vec u|}\right)$ by the unit vector $\frac{\vec u}{|\vec u|}$ in the direction of $\vec u$.

$$\text{proj}_{\vec u} = \vec y = \underbrace{\left(\vec y \cdot \frac{\vec u}{|\vec u|}\right)}_{\text{"Scalar projection of } \vec y \text{ onto } \vec u\text{"}} \underbrace{\frac{\vec u}{|\vec u|}}_{\text{Unit vector in the direction of } \vec u.}$$

In the language of the book

a) Scalar projection of $\vec{y}$ onto $\vec{u}$:

$$\mathrm{Comp}_{\vec{u}}\,\vec{y} = \vec{y}\cdot\frac{\vec{u}}{|\vec{u}|} = \frac{\vec{y}\cdot\vec{u}}{|\vec{u}|}$$

b) Vector projection of $\vec{y}$ onto $\vec{u}$:

$$\mathrm{proj}_{\vec{u}}\,\vec{y} = \left(\frac{\vec{y}\cdot\vec{u}}{\hat{u}}\right)\frac{\vec{u}}{|\vec{u}|} = \frac{y\cdot\vec{u}}{|\vec{u}|^2}\,\vec{u}$$

Exercise # 44 Find scalar and vector projection of $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$ onto $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$

or $\vec{b} = \langle 1,-1,1\rangle$, $\vec{a} = \langle 1,1,1\rangle$

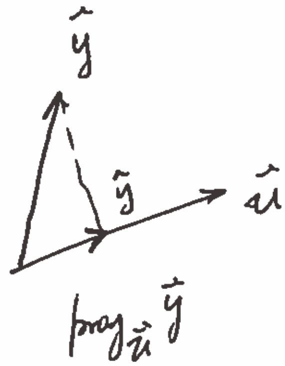
Scalar: $$\mathrm{Comp}_{\vec{a}}\,\vec{b} = \langle 1,-1,1\rangle\cdot\frac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|}$$

$$= \langle 1,-1,1\rangle\cdot\frac{\langle 1,1,1\rangle}{\sqrt{3}} = \frac{1}{\sqrt{3}}(1-1+1) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Vector: $$\mathrm{proj}_{\vec{a}}\,\vec{b} = \frac{\cancel{\sqrt{3}}}{3}\,\frac{\langle 1,1,1\rangle}{\cancel{\sqrt{3}}} = \frac{1}{3}\langle 1,1,1\rangle = \frac{1}{3}(\hat{\imath}+\hat{\jmath}+\hat{k})$$

Find the orthogonal projection of

$\vec{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (Use your linear algebra knowledge)




$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{[1\ -1\ 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{[1\ 1\ 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Vector projection of $\vec{y}$ onto $\vec{u}$.

<u>Alternative</u>

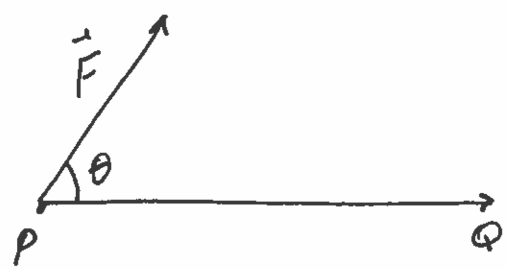
$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{|\vec{u}|^2}\vec{u} = \left(\frac{\vec{y}\cdot\vec{u}}{|\vec{u}|}\right)\frac{\vec{u}}{|\vec{u}|} = \left(\frac{[1\ -1\ 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \left(\frac{1}{\sqrt{3}}\right)\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Unit vector in the direction of $\vec{u}$.

Scalar projection of $\vec{y}$ onto $\vec{u}$

## Work done by a force.




Compute work done by force $\vec{F}$ to move object from P to Q.

**Def:** The work done by force $\vec{F}$ to move object from P to Q is defined to be the product of the component of $\vec{F}$ in the direction of the motion $\vec{PQ}$, and the distance moved.

$$\boxed{W = |\vec{F}|\cos\theta\,|\vec{PQ}| = \vec{F} \bullet \vec{PQ}}$$

**Exercise # 49** $\vec{F} = 8\hat{i} - 6\hat{j} + 9\hat{k}$ N.

Motion from $P(0,10,8)$ to $Q(6,12,20)$ m. along straight line

$\vec{PQ} = \langle 6, 2, 12 \rangle$

$$\Rightarrow \quad W = \vec{F} \cdot \vec{PQ} = \langle 8, -6, 9 \rangle \cdot \langle 6, 2, 12 \rangle = 48 - 12 + 108$$

$$= 144 \text{ Joules.}$$

Section 12.3 Some problems

Find scalar and vector projections of $\vec{b}$ onto $\vec{a}$.

#41) $\vec{a} = \langle 4, 7, -4 \rangle$, $\vec{b} = \langle 3, -1, 1 \rangle$.

Answ:

Scalar:

$$\text{Comp}_{\vec{a}} \vec{b} = \vec{b} \cdot \frac{\vec{a}}{|\vec{a}|} = \langle 3, -1, 1 \rangle \cdot \frac{1}{9} \langle 4, 7, -4 \rangle$$

$$|\vec{a}| = \sqrt{16 + 49 + 16} = \sqrt{81} = 9$$

$$= \frac{12}{9} - \frac{7}{9} - \frac{4}{9} = \frac{1}{9}$$

Vector Projection:

$$\text{proj}_{\vec{a}} \vec{b} = \left( \vec{b} \cdot \frac{\vec{a}}{|\vec{a}|} \right) \frac{\vec{a}}{|\vec{a}|} = \left( \text{Comp}_{\vec{a}} \vec{b} \right) \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{1}{9} \left\langle \frac{4}{9}, \frac{7}{9}, \frac{-4}{9} \right\rangle = \left\langle \frac{4}{81}, \frac{7}{81}, \frac{-4}{81} \right\rangle.$$